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Study of the Reaction $\pi^- p \to \eta\pi^- p$ at 18 GeV/c

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Abstract

Details of the analysis of the $\eta\pi^-$ system studied in the reaction $\pi^- p \to \eta\pi^- p$ at 18 GeV/c are given. Separate analyses for the 2γ and $\pi^+\pi^-\pi^0$ decay modes of the η are consistent. The amplitude analysis indicates the presence of interference between the $a_2(1320)$ and a $J^{PC} = 1^{-+}$ wave between 1.2 and 1.6 GeV/c^2. The phase difference between these waves shows phase motion not attributable solely to the $a_2(1320)$. The data can be fitted by interference between the $a_2(1320)$ and an exotic 1^{-+} resonance with M = $(1370 \pm 16\ ^{+50}_{-30})$ MeV/c^2 and $\Gamma = (385 \pm 40\ ^{+65}_{-105})$ MeV/c^2. These results are compared with those of other experiments.

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I. INTRODUCTION

In a previous publication [1], we presented evidence for an exotic meson produced in the reaction

$$\pi^- p \rightarrow \eta\pi^- p \tag{1}$$

at 18 GeV/c, with the decay mode $\eta \rightarrow \gamma\gamma$. The purpose of this paper is to provide details of that analysis, to discuss additional analyses of that data, and to provide a detailed comparison of our results with those of other experiments. We also provide a comparison of those results with data from our experiment on Reaction (1) but from the $\eta \rightarrow \pi^+\pi^-\pi^0$ decay mode.

The $\eta\pi$ system is particularly interesting in searching for exotic (or non-$q\overline{q}$) mesons because the system has spin (J), parity (P), and charge-conjugation (C) in the sequence $J^{PC} = 0^{++}, 1^{-+}, 2^{++}, 3^{-+}...$ for $L = 0, 1, 2, 3, ...$. (Here L is the orbital angular momentum of the $\eta\pi$ system.) Hence a resonance with an $\eta\pi$ decay mode with odd L is manifestly exotic.[a] Having isospin I=1, such a resonance could not be a glueball $(2g, 3g, \ldots)$, but it could be a hybrid $(q\overline{q}g)$ or a multiquark $(q\overline{q}q\overline{q})$ state.

A. Models

Properties of hybrids and multiquark mesons have been discussed in the framework of various models [2–9]. Calculations based upon the MIT bag model predict [3,4] that an $I = 1$, 1^{-+} hybrid $(q\overline{q}g)$ will have a mass near 1.4 GeV/c^2. On the other hand, the flux-tube model [5,6] predicts the mass of the lowest-lying hybrid state to be around 1.8 GeV/c^2. Characteristics of bag-model S-wave multiquark states (which would have $J^P = 0^+$, 1^+ or 2^+) have been discussed [7] but those for a 1^- state have not. Recently, an analysis of the

[a] A $q\overline{q}$ meson with orbital angular momentum ℓ and total spin s must have $P = (-1)^{\ell+1}$ and $C = (-1)^{\ell+s}$. A resonance with $J^{PC} = 0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, 3^{-+} \cdots$ does not satisfy these conditions and must be exotic.

multiquark hybrids has been carried out, based on the diquark cluster model [8]; this model predicts a lowest-lying isovector $J^{PC} = 1^{-+}$ state at 1.39 GeV with a very narrow width ($\simeq$ 8 MeV). Ultimately, the most reliable estimate of the masses is expected to come from lattice-gauge calculations; the latest mass estimates for a 1^{-+} hybrid are in the range 1.7 to 2.1 GeV [9].

B. Previous Experiments

Production of $\eta\pi$ events with $J^{PC} = 1^{-+}$ has been observed in several experiments [10–13]. Alde et al. [10], in a study of $\pi^- p$ interactions at 100 GeV/c at CERN (the GAMS experiment), claimed to observe a 1^{-+} state in the $\eta\pi^0$ system at 1.4 GeV/c^2 produced via *unnatural* parity exchange (the P_0 partial wave—the naming convention is discussed below). This result was later brought into question [14]. Aoyagi et al. [11], in a $\pi^- p$ experiment at 6.3 GeV/c at KEK, observed a rather narrow enhancement in the $\eta\pi^-$ system at 1.3 GeV/c^2 in the *natural* parity exchange 1^{-+} spectrum (P_+). Beladidze et al. [12], in the VES experiment at IHEP, ($\pi^- N$ interactions at 37 GeV/c) also reported a P_+ signal in the $\eta\pi^-$ state, but their signal was broader and had a significantly different phase variation from that of the KEK experiment. While the phase difference between the P_+ and D_+ waves was independent of $\eta\pi$ mass in the KEK analysis, that phase difference did show significant mass dependence in the VES analysis. Finally, the Crystal Barrel experiment at CERN has recently reported [13] on the presence of signals in $N\overline{N}$ annihilations in both the charged and neutral $\eta\pi$ systems. In this paper, we compare the results of our analysis with those of references [11–13].

II. EXPERIMENTAL DETAILS

A. E852 Apparatus

Our data sample was collected in the first data run of E852 at the Alternating Gradient Synchrotron (AGS) at Brookhaven National Laboratory (BNL) with the Multi-Particle Spectrometer (MPS) [15] augmented by additional detectors. A diagram of the experimental apparatus is shown in Fig. 1. A tagged π^- beam of momentum 18 GeV/c was incident on a one-foot long liquid hydrogen target at the center of the MPS magnet. The target was surrounded by a four-layer cylindrical drift chamber (TCYL) [16] used to trigger on the proton recoil of Reaction (1), and a 198-element cylindrical thallium-doped cesium iodide array (CsI) [17] capable of rejecting events with wide-angle photons. The downstream half of the magnet was equipped with six drift chamber modules (DC1-6) [18], each consisting of seven planes, used for charged-particle tracking. Interspersed among these were three proportional wire chambers (TPX1-3) to allow triggering on the multiplicity of forward tracks; a window-frame lead scintillator photon veto counter (DEA) to ensure photon hermeticity; a forward scintillation counter (CPVB) to veto forward charged tracks for neutral triggers; and a window-frame scintillation counter (CPVC) to identify charged particles entering the DEA. Beyond the magnet were a newly-built drift chamber (TDX4) consisting of two x-planes; two scintillation counters (BV and EV) to veto non-interacting beam tracks and elastic scatters respectively; and a 3045-element lead glass electromagnetic calorimeter (LGD) [19] to detect forward photons. Further details are given elsewhere [20].

B. Trigger

The trigger for Reaction (1) required a recoil charged particle in the TCYL detector and one charged particle traversing each of the first two TPX chambers. In addition, an electronic algorithm coupling energy and position information in the LGD calorimeter [19] (an "effective-mass" trigger) was utilized for the purpose of enhancing the fraction of η's relative to π^0's in the sample. A total of 47 million triggers of this type were recorded.

C. Event Reconstruction and Selection

Of the 47 million triggers, 47,200 events were reconstructed which were consistent with Reaction (1). These were selected by requiring:

1. two photons reconstructed in the LGD;
2. one forward track reconstructed in DC1-6;
3. one recoil track reconstructed in TCYL;
4. a common vertex, in a target fiducial volume, reconstructed from the charged tracks and the beam track;
5. no photon energy deposited in the DEA detector or outside the fiducial volume of the LGD;
6. a difference in azimuth of $\leq 20°$ between the reconstructed TCYL track and the missing momentum vector;
7. $\leq$ 20 MeV of energy deposit (beyond that associated with the TCYL track) in the CsI detector to require a proton recoil (analysis was carried out with and without this cut); and
8. a SQUAW [21] kinematic fit (requiring energy and momentum conservation) to Reaction (1) with a confidence level $> 10\%$.

D. Experimental Acceptance

The experimental acceptance is determined by a Monte Carlo method. Events are generated [22] with isotropic angular distributions in the Gottfried-Jackson (GJ) frame. (The GJ frame is a rest frame of the $\eta\pi^-$ system in which the z-axis is in the direction of the beam momentum, and the y-axis is in the direction of the vector cross-product of the target and recoil momenta.) After adding detector simulation using GEANT [23], the Monte Carlo event sample is subjected to the same event-selection cuts and run through the same analysis as the data. A second method (called SAGEN) which did not use GEANT was employed as well. This method also used SAGE as the event generator, but instead of using GEANT, a geometrical acceptance was used. This second method allowed acceptances to be calculated much more quickly. The amplitude analysis results discussed below obtained from the two methods were found to be consistent with each other. The average acceptances shown in Figs. 2-4 are based on the SAGEN method.

The average acceptance as a function of $\eta\pi^-$ effective mass is shown in Fig. 2. The average acceptance decreases by about a factor of two over the effective mass region from 1.0 to 2.0 GeV/c^2 (Average acceptances are calculated for peripheral production and isotropic decay.)

Shown in Figs. 3 and 4 is the acceptance as a function of $cos\theta$ and of ϕ for various values of the $\eta\pi^-$ effective mass. Here $cos\theta$ and of ϕ are are the polar and azimuthal angles measured in the GJ frame. The polar angle is the angle between the beam direction and the η direction in this frame. The inefficiency in the backward direction corresponds to slow η's and fast π^-'s in the lab. The slow η's lead to slow γ's which are often produced at wide angles and thus miss the LGD. The acceptance in ϕ is relatively uniform.

Finally, shown in Fig. 5 is the average acceptance (GEANT-based) as a function of $|t'| = |t - t_{\min}|$, where t is the the four-momentum-transfer between the initial- and final-state protons. The dramatic decrease in acceptance below about $|t'| = 0.08(GeV/c)^2$ is due to a trigger requirement. In particular, since we require the presence of a recoil proton in TCYL, the trigger cannot be satisfied if the proton stops in the hydrogen target.

E. Background Studies

Shown in Fig. 6 is the 2γ effective mass distribution for events in the $a_2(1320)$ mass region. The central cross-hatched region in Fig. 6 shows the events which remain after the SQUAW-based kinematic-fitting cuts. The width of the η peak ($\sigma \approx .03$ GeV/c^2) is consistent with that expected from Monte Carlo studies. Two methods have been used to study the background in our sample. Method 1 used the shaded sidebands of Fig. 6 and allows us to study the non-η background in the data.

In Fig. 7 is shown the missing-mass squared distribution for the data sample before kinematic fitting. The shaded area shows the events which remain after kinematic fitting. Recall that one expects the distribution for good events for Reaction 1 to peak at the square of the proton mass or at a value of 0.88 GeV/c^2 .

A scatterplot of the 2γ effective mass versus the missing-mass squared is shown in Fig. 8. Background studies using our Method 2 takes as the background estimator a region surrounding the central signal region seen here instead of using the sidebands of Fig. 6. In this way, background events of both the non-η type and of the type with missing π^0's are taken into account. (The background is estimated using the region included within the outer elliptical area of Fig. 8 but not within the middle elliptical region.)

Shown in Fig. 9 is the effective-mass distribution of the background, estimated using Method 1. In this figure are shown the effective-mass distribution for each side-band region as well as the summed distribution for the background regions. Because the background regions have different thresholds, one higher than the signal region and one lower than the signal region, the histograms are shifted by an appropriate amount (so that their thresholds match that of the signal region) before summing.

In Fig. 10 is shown the polar angular distribution of the background events from Method 1 in the $a_2(1320)$ effective-mass region. Both uncorrected and corrected angular distributions are shown. This angular distribution is consistent with isotropy although there is some tendency for the distribution to have an excess below the region $\cos\theta < -0.5$.

III. GENERAL FEATURES OF THE DATA

The $a_2(1320)$ is the dominant feature of the $\eta\pi^-$ effective-mass spectrum shown in Fig. 11. The background, which is shown shaded in the figure, is estimated from Method 2 above, and is approximately 7% at 1.2 GeV/c^2, falling to 1% at 1.3 GeV/c^2.

The acceptance-corrected distribution of $|t'| = |t - t_{\mathrm{min}}|$ is shown for $|t'| > 0.08(\mathrm{GeV/c})^2$ in Fig. 12. (Our acceptance is quite low below 0.08 $(\mathrm{GeV/c})^2$ as discussed in section II D.) Here t is the the four-momentum-transfer between the target proton and the outgoing proton and t_{min} is the minimum value for this quantity for a given $\eta\pi$ effective mass. The shape of this distribution is consistent with previous experiments and has been shown to be consistent with natural-parity exchange production in Regge-pole phenomenology [24,25].

The acceptance-corrected[b] distribution of $\cos\theta$, the cosine of the angle between the η and the beam track in the GJ frame, is shown in Fig. 13 for various ranges of $\eta\pi^-$ effective mass. The presence of a significant forward-backward asymmetry in $\cos\theta$ is obvious.

The forward-backward asymmetry in $\cos\theta$ is plotted as a function of $\eta\pi$ effective mass in Fig. 14. Here, the asymmetry is defined as $(F-B)/(F+B)$ where $F(B)$ is the number of events in the mass bin with the η decaying forward (backward) in the GJ frame. For this figure, the asymmetry was calculated for events in the higher-acceptance region with $|\cos\theta| < 0.8$.[c] The asymmetry is large, statistically significant and mass dependent. Since the presence of only even values of L would yield a symmetric distribution in $\cos\theta$, the observed asymmetry requires that odd-L partial waves be present and that they interfere with even-L partial waves to describe the data.

The azimuthal angular distribution as a function of $\eta\pi^-$ effective mass is shown in Fig. 15. The observed structure has a clear $\sin\phi$ component which indicates the presence of $m = 1$

[b] For illustration purposes, the acceptance correction is calculated here for isotropic decay of the $\eta\pi$ system. The acceptance correction used in the amplitude analysis discussed below is based upon the observed decay angular distribution.

[c] The asymmetry function was plotted for various ranges of the decay angle and the presence of a strong asymmetry was noted in all cases.

natural-parity-exchange waves in the production process. (See the discussion in Section IV below.)

Shown in Fig. 16 are the $\pi^- p$ and ηp effective-mass distributions for the data sample. It is important to note that the absence of baryon isobar production is required for the assumptions of our PWA to be valid. There is at most a very small amount of isobar production in the region $M(\pi p) < 2.0$ GeV/c^2 in Fig. 16a and none in Fig. 16b. The amplitude analysis described in Section IV was checked to insure that isobar production did not effect our results. This was done by redoing the analysis after requiring $M(\pi p) > 2.0$ GeV/c^2 . The results were not effected by this cut.

IV. PARTIAL-WAVE ANALYSIS

A. Procedure

A partial-wave analysis (PWA) [26,27] based on the extended maximum likelihood method has been used to study the spin-parity structure of the $\eta\pi^-$ system. In Appendix A, a brief description of the formalism is given as are the relationships between the the partial wave amplitudes (assuming $L \leq 2$) and the moments of the angular distribution. The technique of the extended maximum likelihood analysis is covered in Appendix B, where the interplay of the experimental moments and the acceptance is described. (The experimental acceptance is incorporated into the PWA by using the accepted Monte Carlo events described above to calculate normalization integrals – see ref. [26]).

The partial waves are parameterized in terms of the quantum numbers J^{PC} as well as m, the *absolute value* of the angular momentum projection, and the reflectivity ϵ [28]. In our naming convention, a letter indicates the angular momentum of the partial wave in standard spectroscopic notation, while a subscript of 0 means $m = 0$, $\epsilon = -1$, and a subscript of $+(-)$ means $m = 1$, $\epsilon = +1(-1)$. Thus, S_0 denotes the partial wave having $J^{PC}m^\epsilon = 0^{++}0^-$, while P_- signifies $1^{-+}1^-$, D_+ means $2^{++}1^+$, and so on.

We consider partial waves with $m \leq 1$ in our analysis. This assumption is true, of course, in the limit of $-t = 0$, since the nucleon helicities give rise to the states with $m = 0$ or $m = \pm 1$ only. But this assumption can be dealt with—experimentally—since the moments $H(LM)$ with $M = 3$ or $M = 4$ can be checked, to see how important the states $|\ell\, m\rangle$ are in the data with $|m| \geq 2$. This has been done with our data. The moments $H(33)$, $H(43)$ and $H(44)$ are all small in the $a_2(1320)$ region, and a fit including $|22\rangle$ shows only a very small amount of this wave and is very broad. In particular, it does not affect our P_+ wave!

We also assume that the production spin-density matrix has rank one. This assumption is discussed in Appendix C.

Goodness-of-fit is determined by calculation of a χ^2 from comparison of the experimental moments with those predicted by the results of the PWA fit. A systematic study has been performed to determine the effect on goodness-of-fit of adding and subtracting partial waves of $J \leq 2$ and $M \leq 1$. We find that although no significant structure is seen in the waves of negative reflectivity (see below), their presence in the PWA fit results in a significant improvement in goodness-of-fit compared to a fit which includes only the dominant positive-reflectivity partial waves. We have also performed fits including partial waves with $J = 3$ and $J = 4$. Contributions from these partial waves are found to be insignificant for $M(\eta\pi^-) < 1.8$ GeV/c^2. Thus, PWA fits shown or referred to in this letter include all partial waves with $J \leq 2$ and $m \leq 1$ (i.e. S_0, P_0, P_-, D_0, D_-, P_+, and D_+). A non-interfering, isotropic background term of fixed magnitude determined as described by Method 2 in Section II D is used.

B. Results

The results of the PWA fit of 38,200 events in the range $0.98 < M(\eta\pi^-) < 1.82$ GeV/c^2 and $0.10 < |t| < 0.95$ GeV^2 are shown in Figs. 17 and 18. In Fig. 17 the acceptance-corrected numbers of events predicted by the PWA fit for the D_+ and P_+ intensities and the phase difference between these amplitudes, $\Delta\Phi$, are shown as a function of $M(\eta\pi^-)$. (The smooth curves shown in this figure are discussed below in Section V B. There are eight ambiguous solutions in the fit [27,29,30]. These solutions are mathematically discrete but with equal likelihoods – that is, they predict exactly the same values for the angular observables. We show the range of fitted values for these ambiguous solutions in the vertical rectangular bar at each mass bin, and the maximum extent of their errors is shown as the error bar. [d]

The $a_2(1320)$ is clearly observed in the D_+ partial wave (Fig. 17a). A broad peak is seen in the P_+ wave at about 1.4 GeV/c^2 (Fig. 17b). The phase difference $\Delta\Phi$ increases through the $a_2(1320)$ region, and then decreases above about 1.5 GeV/c^2 (Fig. 17c). This phase behavior will allow us to study the nature of the P_+ wave.

Shown in Fig. 18 are the fitted intensities for waves which are produced by negative-reflectivity (or unnatural-parity) exchange. The predicted numbers of events for these waves are generally small and are all consistent with zero above about 1.3 GeV/c^2. Although there is some non-zero contribution from the D_- and (especially) the S_0 waves below this region, the uncertainties and ambiguity ranges associated with these waves make it impossible to do a definitive study of them to determine their nature.

It is important to note that the forward-backward asymmetry noted earlier is due to interference in the natural-parity exchange sector rather than to the unnatural-parity exchange waves. This is illustrated in Fig. 19 which shows the predicted asymmetry separately for the natural and unnatural parity exchange waves. It is clear that the asymmetry due to the unnatural-parity waves is about an order of magnitude less than that due to the natural-parity waves. Also shown in Fig. 19 is the comparison of the asymmetry present in the data with that predicted by the fit. The fit clearly does an excellent job in representing the data points.

The unnormalized spherical harmonic moments $H(LM)$ and their prediction from the PWA fit as a function of mass are shown in Fig. 20. Here $H(LM) = \sum_{i=1}^{N} Y_L^M(\theta_i, \phi_i)$ (N being the number of events in a given bin of $M(\eta\pi^-)$). The relationships between the moments and the amplitudes is given in Appendix A, Eqs. A27. An examination of the $H(30)$, $H(32)$,

[d] These rectangular bars are quite small and thus not apparent for the D_+ intensity, but are quite clear for the P_+ intensity and the phase difference distributions.

$H(40)$ and $H(42)$ moments along with a comparison with Eqs. A27 shows that the D_+ amplitude dominates and demonstrates clearly that the P_+ partial wave is required for the PWA fit to describe the experimental moments. These moments cannot be described solely by the combination of the D_+ partial wave and experimental acceptance.

The change in −log(Likelihood) brought about by variation of the predicted number of events for the P_+ partial wave for the $1.30 < M(\eta\pi^-) < 1.34$ GeV/c^2 bin is shown for all the ambiguous solutions in Fig. 21. The number of predicted P_+ events from the PWA fit ranges from 330 events to 530 events with typical errors of 280 events. (A change in −log(Likelihood) of 0.5 corresponds to one standard deviation.) For all solutions, the liklihood function gets so bad below 100 events that the P_+ wave is clearly required to fit the data. Thus the observed variation in −log(Likelihood) further demonstrates that the P_+ partial wave is required to describe our data.

C. Systematic Studies

PWA fits have been performed in various regions of $|t|$. A fit of 20,000 events in the range $0.0 < |t| < 0.25$ GeV^2 yields results consistent with those of the fit shown. The $a_2(1320)$ is still clearly observed in the D_+ partial wave as is the structure in the P_+ wave. In a fit of 1000 events in the range $0.95 < |t| < 10.0$ GeV^2, no significant structure is seen in any partial wave other than D_+, in which the $a_2(1320)$ is again observed. The lack of statistics at large $|t|$ precludes us from drawing further conclusions from this observation.

A PWA fit has been carried out excluding those events with $|cos\theta_{GJ}| > 0.8$ (the region in which experimental acceptance is poorest). The results of this fit are consistent with those shown above.

To test for Δ and N^* contamination, a fit has been done in which events with $M(\pi^- p) <$ 2.0 GeV/c^2 are excluded. Again, this cut is seen to have no qualitative effect on the PWA results.

Fits were also carried out on Monte Carlo events generated with a pure D_+ wave to determine whether structure in the P_+ wave could be artificially induced by acceptance effects, resolution, or statistical fluctuations. Shown in Fig. 22 are the results of such a fit. We do find that a P_+ wave can be induced by such effects. This 'leakage' leads to a P_+ wave that: (1) mimics the generated D_+ intensity (and in our case would therefore have the shape of the $a_2(1320)$); and (2) has a phase difference $\Delta\Phi$ that is independent of mass. Neither property is present in our study and we conclude that the P_+ structure which we observe is not due to 'leakage'.

Fits have been performed allowing $L = 3$ and $L = 4$. We find that these waves are negligible in the region below $M(\eta\pi)$ of about 1.7-1.8 GeV/c^2.

The data have been fit using different parametrizations of the background. The background has been set at fixed values determined from the two different background estimates discussed previously. In another fit, the background has been set to zero. And finally, a fit was performed allowing the background level to be a free parameter. Although the negative reflectivity waves do change somewhat for different treatments of the background [e], the results for the D_+ and P_+ waves are very stable. In particular, the behavior of the relative phase between the P_+ and the D_+ waves does not change substantively in any of these checks.

D. Comparison with Previous Experiments

These results for the P_+ and D_+ intensities and their phase difference are quite consistent with the VES results [12] as can be seen in Fig. 23. In particular, the behavior of the shape of the phase difference is virtually identical to that reported by VES. [f] This is particularly noteworthy since, as will be seen below, it is this phase difference which allows us to draw

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[e] Since the background and the S_0 wave are both isotropic, the fitting program cannot distinguish easily between them.

[f] The magnitude of the phase difference is shifted by about 20° relative to that of VES. A production phase shift would not be unexpected because of the differing energies and targets in the two experiments.

conclusions regarding the nature of the P_+ wave.

Our results are compared with those of the KEK experiment [11] in Fig. 24. In this case, it is clear that the two results differ. The KEK results have a P-wave intensity which is narrower (much like the $a_2(1320)$ shape) and a P-D phase difference which is constant – results which are similar to those shown in our Monte Carlo study where the P_+ wave is not present in the sample but is induced by "leakage".

E. Comparison with $\eta \to \pi^+\pi^-\pi^0$ Data Sample

A second data set in another topological class (with two additional charged particles in the final state) has been used to study Reaction (1) with the decay mode $\eta \to \pi^+\pi^-\pi^0$. This data sample has significantly different acceptance than the $\eta \to 2\gamma$ sample and the systematics are generally different as well.

Shown in Fig. 25 is the $\pi^+\pi^-\pi^0$ effective mass distribution from this data set. There is a clear η peak as well as a strong peak in the ω region. After kinematic fitting, a sample of 2,235 events was obtained which was consistent with Reaction 1 with $\eta \to \pi^+\pi^-\pi^0$ and on which an amplitude analysis was carried out. Fig. 26 shows the effective mass distribution for this sample of events. As expected, the $a_2(1320)$ dominates this spectrum.

Although the data sample is very small for this topology, we have carried out an amplitude analysis in order to compare with the primary $\eta \to 2\gamma$ analysis. Results of the analysis are shown in Fig. 27 where we compare the shapes of the P_+ intensities for the two data sets as well as the $P_+ - D_+$ phase differences. Despite the rather large statistical uncertainties, there is remarkable agreement between these distributions.

V. MASS-DEPENDENT FIT

In an attempt to understand the nature of the P_+ wave observed in our experiment, we have carried out a *mass-dependent* fit to the results of the mass-independent amplitude analysis. The fit has been carried out in the $\eta\pi$ mass range from 1.1 to 1.6 GeV/c^2. In this fit, we have assumed that the D_+-wave and the P_+-wave decay amplitudes are resonant and have used relativistic Breit-Wigner forms for these amplitudes.

A. Procedure

We shall use a shorthand notation w to stand for the $\eta\pi$ mass, i.e. $w = M(\eta\pi^-)$. Representing the mass-dependent amplitudes for D_+ and P_+ as $V_\ell(w)$ for $\ell = 2$ and 1, we may write

$$V_\ell(w) = \mathrm{e}^{i\,\phi_\ell}\,\Delta_\ell(w)\,B_\ell(q)\;[a_\ell + b_\ell(w - w_\ell^0) + c_\ell(w - w_\ell^0)^2]^{1/2} \tag{2}$$

where q is the $\eta\pi$ breakup momentum at mass w. Here ϕ_ℓ is the production phase (mass independent[g]), associated with a wave ℓ. The quantities $\Delta_\ell(w)$ and $B_\ell(q)$ are the standard relativistic Breit-Wigner form and the barrier factor, respectively, and are given below. The square-root factor has been introduced primarily to take into account possible deviations from the standard Breit-Wigner form, away from the resonance mass (denoted by w_ℓ^0). The overall normalization of a wave is governed by a_ℓ, while the constants b_ℓ and c_ℓ allow for deviations in the mass spectra from the Breit-Wigner form. The constants a_ℓ, b_ℓ and c_ℓ are all real, so that the square-root factor does not effect the rapidly varying phase implied by the standard Breit-Wigner form.[h]

The barrier functions [31], which are real, are given by

[g] We have tried a linear dependence in mass for the phase; the fits did *not* require it.

[h] For a Breit-Wigner form with a constant width, the phase rises 90 degrees over one full width centered at the resonance mass.

$$
\begin{aligned}
B_0(q) &= 1 \\
B_1(q) &= \left[\frac{z}{z+1}\right]^{1/2} \\
B_2(q) &= \left[\frac{z^2}{(z-3)^2+9z}\right]^{1/2} \\
B_3(q) &= \left[\frac{z^3}{z(z-15)^2+9(2z-5)}\right]^{1/2} \\
B_4(q) &= \left[\frac{z^4}{(z^2-45z+105)^2+25z(2z-21)^2}\right]^{1/2}
\end{aligned} \tag{3}
$$

where $z = (q/q_{_R})^2$ and $q_{_R} = 0.1973$ GeV/c corresponding to 1 fermi. Note that $B_\ell(p) \Rightarrow\ 1$ as $q \Rightarrow\ \infty$.

The relativistic Breit-Wigner functions can be written

$$
\Delta_\ell(w) = \left[\frac{\Gamma_\ell^0}{\Gamma_\ell(w)}\right] \mathrm{e}^{i\,\delta_\ell(w)}\ \sin\delta_\ell(w) \tag{4}
$$

where Γ_ℓ^0 is the nominal width (mass independent) and $\Gamma_\ell(w)$ is the mass-dependent width given by

$$
\Gamma_\ell(w) = \Gamma_\ell^0 \left(\frac{w_\ell^0}{w}\right)\left(\frac{q}{q_\ell^0}\right)\left[\frac{B_\ell(q)}{B_\ell(q_\ell^0)}\right]^2 \tag{5}
$$

where q_ℓ^0 is the breakup momentum evaluated at $w = w_\ell^0$. The mass-dependent phase shift $\delta_\ell(w)$ is given by

$$
\cot\delta_\ell(w) = \left[\frac{w_\ell^0}{\Gamma_\ell(w)}\right]\left[1-\left(\frac{w}{w_\ell^0}\right)^2\right] \tag{6}
$$

and the overall phase for the ℓ-wave amplitude is

$$
\Phi_\ell = \phi_\ell + \delta_\ell(w). \tag{7}
$$

Since we are dealing with two waves P_+ and the D_+, we can only measure $\phi = \phi_2 - \phi_1$. Then the phase difference being measured experimentally, corresponds to

$$
\Delta\Phi = \Phi_2 - \Phi_1 = \phi + \delta_2(w) - \delta_1(w) \tag{8}
$$

Finally, the experimental mass distribution for each wave ℓ is given by

$$\frac{\mathrm{d}\sigma_\ell}{\mathrm{d}w} = |V_\ell(w)|^2\, pq \tag{9}$$

where pq is the phase-space factor for which p is the breakup momentum in the overall center-of-mass frame of the $\eta\pi^-$ system (or of the final-state proton) in Reaction ((1)). Since the problem here is for a given $\sqrt{s}$, all other relevant factors, including that of the beam flux, have been obsorbed into the amplitude itself, i.e. the constants a_ℓ, b_ℓ and c_ℓ.

The input quantities to the fit included, in each mass bin: the P_+-wave intensity; the D_+-wave intensity; and the phase difference $\Delta\Phi$ (the relevant formulas are given in (8) and (9)). Each of these quantities was taken with its error (including correlations) from the result of the amplitude analysis. One can view this fit as a test of the hypothesis that the correlation between the fitted P-wave intensity and its phase (as a function of mass) can be fit with a resonant Breit-Wigner amplitude.

We find that the fit does not improve significantly when the P_+ wave is modified from the Breit-Wigner form, and hence set b_1 and $c_1 = 0$ for the final fit. We also note that the magnitude of the quantities b_2 and c_2 in the final fit correspond to a small deviation of the D_+-wave intensity of the order of 1%.

B. Results

Results of the fit are shown as the smooth curves in Fig. 17a, b, and c. The mass and width of the $J^{PC} = 2^{++}$ state (Fig. 17a) are (1317 ±1 ±2) MeV/c^2 and (127 ±2 ±2) MeV/c^2 respectively [32]. (The first error given is statistical and the second is systematic.) The mass and width of the $J^{PC} = 1^{-+}$ state as shown in Fig. 17b are (1370 ±16 $^{+50}_{-30}$) MeV/c^2 and (385 ±40 $^{+65}_{-105}$) MeV/c^2 respectively. Shown in Fig. 17d are the Breit-Wigner phase dependences for the $a_2(1320)$ (line 1) and the P_+ waves (line 2); the fitted $D_+$$-$$P_+$ production phase difference (line 3); and the fitted $D_+$$-$$P_+$ phase difference (line 4). (Line 4, which is identical to the fitted curve shown in Fig. 17c, is obtained as line 1 $-$ line 2 + line 3.)

The systematic errors have been determined from consideration of the range of solutions possible because of the ambiguous solutions in the PWA. Since there are 8 ambiguous solutions per mass bin and we are fitting over 12 mass bins, it is clearly impossible to try all 8^{12} possible combinations. Instead, we have fit some 10^5 combinations where the values to be fitted in each mass bin have been chosen at random from among the 8 ambiguous PWA solutions. The resulting fits generally clump into a group with reasonable values of $\chi^2/dof(\leq 2)$ and into a group with poor values. The systematic errors on the mass and width given above are taken from the extremes observed for the solutions with reasonable values of χ^2/dof. The central values quoted above are taken from a fit which uses the average values of the input parameters in each bin.

In order to determine the sensitivity of our result to the exact function being used to fit the D-wave intensity distribution we have redone the fit using two other hypotheses. First we have performed a fit in which the mass-dependent amplitude is given by Eq.(2), but with $b_2 = c_2 = 0$. Second, we have taken $b_2 = c_2 = 0$ in Eq.(2) and also replaced the Blatt-Weisskopf barrier functions for each wave by the factor q^ℓ. Although the resulting fits are poorer in quality, we find that the parameters of the fit do not change significantly compared to the systematic uncertainty described above.

The fit to the resonance hypothesis has a χ^2/dof of 1.49. The fact that the production phase difference can be fit by a mass-independent constant (of 0.6 rad) is consistent with Regge-pole phenomenology[i] in the absence of final-state interactions. If one fits the data to a non-resonant (constant phase) P_+ wave, and also postulates a Gaussian intensity distribution for the P_+ wave, one obtains a very poor fit with a χ^2/dof of 7.08. Finally if one allows a *mass-dependent production phase*, a χ^2/dof of 1.55 is obtained for the non-resonant hypothesis — but the production phase must have a very rapid variation with mass.[j] Such a phase variation cannot be excluded, but is not expected for any known model. Note that for this

[i] The signature factor and the residue functions are at most t-dependent (not mass dependent) (see ref. [24]).

[j] The fit requires a linear production phase difference with a slope of -4.3 rad/GeV.

non-resonant hypothesis one must have a separate hypothesis for the observed structure in the P_+ intensity — a structure which is explained naturally by the resonance hypothesis.

Our fitted parameters are compared with values recently reported by the Crystal Barrel experiment [13] in Table I. That experiment reports that a $J^{PC} = 1^{-+}$ resonance in the $\eta\pi$ channel is required to fit their data in the annihilation channel $\overline{p}n \to \pi^-\pi^0\eta$. Their fitted parameters are very consistent with those determined from our mass-dependent analysis. The fact that similar properties are determined from these very different reactions lends further credence to the hypothesis that the results obtained are due to physical effects rather than to some artifact of the analysis.

VI. SUMMARY AND CONCLUSIONS

In this paper, we have discussed the details of the amplitude analysis of data from Reaction 1. Interference between D-wave and P-wave amplitudes produced with natural parity exchange is required in order to explain the data. Using this interference, we have shown that the P-wave phase has a rapid variation with mass and that this phase variation coupled with the fitted P-wave intensity distribution is well-fitted by a Breit-Wigner resonance with mass and width of ($1370 \pm 16\ {}^{+50}_{-30}$) MeV/$c^2$ and ($385 \pm 40\ {}^{+65}_{-105}$) MeV/$c^2$ respectively. We conclude that there is credible evidence for the production of a $J^{PC} = 1^{-+}$ exotic meson.

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APPENDIX A: PARTIAL-WAVE FORMULAS

In this appendix, the angular distributions are derived for the $\eta\pi^-$ system produced in Reaction (1). The distributions are given both in terms of the moments and the amplitudes in the reflectivity basis. For a system consisting of S, P and D waves, explicit formulas for the moments as functions of the partial waves are also given.

In the Gottfried-Jackson (GJ) frame, the amplitudes may be expanded in terms of the partial waves for the $\eta\pi^-$ system:

$$U_k(\Omega) = \sum_{\ell m} V_{\ell m k} A_{\ell m}(\Omega) \tag{A1}$$

where $V_{\ell m k}$ stands for the production amplitude for a state $|\ell m\rangle$ and k represents the spin degrees of freedom for the initial and final nucleons ($k = 1, 2$ for spin-flip and spin-nonflip amplitudes). $A_{\ell m}(\Omega)$ is the decay amplitude given by

$$A_{\ell m}(\Omega) = \sqrt{\frac{2\ell+1}{4\pi}}\, D^{\ell\,*}_{m0}(\phi, \theta, 0) = Y_\ell^m(\Omega) \tag{A2}$$

where the angles $\Omega = (\theta, \phi)$ describe the direction of the η in the GJ frame. It is noted, in passing, that the small d-function implicit in (A2) is related to the associated Legendre polynomial via

$$d^\ell_{m0}(\theta) = (-)^m \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta) \tag{A3}$$

The angular distribution is given by

$$I(\Omega) = \sum_k |U_k(\Omega)|^2 \tag{A4}$$

It should be emphasized that the nucleon helicities are external entities and the summation on k is applied to the absolute square of the amplitudes. A complete study of the $\eta\pi^-$ system requires four variables: $M(\eta\pi^-)$, $-t$ and the two angles in Ω. The distribution (A4) is therefore to be applied to a given bin of $M(\eta\pi^-)$ and of $-t$.

The angular distribution may be expanded in terms of the moments $H(LM)$ via

$$I(\Omega) = \sum_{LM} \left(\frac{2L+1}{4\pi}\right) H(LM) D^{L\,*}_{M0}(\phi, \theta, 0) \tag{A5}$$

where

$$H(LM) = \sum_{\substack{\ell m \\ \ell' m'}} \left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2} \rho^{\ell\ell'}_{mm'} (\ell' m' LM|\ell m)(\ell' 0 L0|\ell 0) \tag{A6}$$

where ρ is the spin-density matrix given by

$$\rho^{\ell\ell'}_{mm'} = \sum_k V_{\ell m k} V^*_{\ell' m' k} \tag{A7}$$

It is seen that the moments $H(LM)$ are measurable quantities since

$$H(LM) = \int \mathrm{d}\Omega \; I(\Omega) \, D^L_{M0}(\phi, \theta, 0) \tag{A8}$$

The normalization integral is

$$H(00) = \int \mathrm{d}\Omega \; I(\Omega) \tag{A9}$$

The symmetry relations for the H's are well-known. From the hermiticity of ρ, one gets

$$H^*(LM) = (-)^M \, H(L - M) \tag{A10}$$

and, from parity conservation in the production process, one finds

$$H(LM) = (-)^M \, H(L - M) \tag{A11}$$

These show that the H's are real.

Parity conservation in the production process can be treated with the reflection operator which preserves all the relevant momenta in the S-matrix and act directly on the rest states of the particles involved. It is important to remember that the coordinate system is always defined with the y-axis along the production normal. In this case the reflection operator is simply the parity operator followed by a rotation by π around the y-axis.

The eigenstates of this reflection operator are

$$|\epsilon\ell m\rangle = \theta(m)\Big\{|\ell m\rangle - \epsilon(-)^m|\ell - m\rangle\Big\} \tag{A12}$$

where

$$\begin{aligned} \theta(m) &= \frac{1}{\sqrt{2}}, \quad m > 0 \\ &= \frac{1}{2}, \quad m = 0 \\ &= 0, \quad m < 0 \end{aligned} \tag{A13}$$

For positive reflectivity, the $m = 0$ states are not allowed, i.e.

$$|\epsilon\ell 0\rangle = 0, \quad \text{if} \quad \epsilon = + \tag{A14}$$

The reflectivity quantum number ϵ has been defined so that it coincides with the naturality of the exchanged particle in Reaction (1). One can prove this by noting that the meson production vertex is in reality a time-reversed process in which a state of arbitrary spin-parity decays into a pion (the beam) and a particle of a given naturality (the exchanged particle)

$$J^{\eta_J} \to s^{\eta_s} + \pi \tag{A15}$$

where η's stand for intrinsic parities. The helicity-coupling amplitude F^J for this decay [33] is

$$A_p^J(M) \propto F_\lambda^J \; D_{M\,\lambda}^{J\,*}(\phi_p, \theta_p, 0) \tag{A16}$$

where λ is the helicity of the exchanged particle and the subscript p stands for the 'production' variables. M is the z-component of spin J in a J rest frame. From parity conservation in the decay, one finds

$$F^J_\lambda = -F^J_{-\lambda} \tag{A17}$$

where one has used the relationships $\eta_J = (-)^J$ (true for two-pseudoscalar systems) and $\eta_s = (-)^s$ (natural-parity exchange). The formula shows that the helicity-coupling amplitude F^J is zero if λ is zero. Since angular momentum is conserved, its decay into two spinless particles in the final state cannot have $M = 0$ along the beam direction (the GJ rest system), i.e. the D^J-function is zero unless $M = \lambda$, if $\theta_p = \phi_p = 0$. Finally, one may identify J with ℓ and M with m, which proves (A14).

The modified D-functions in the reflectivity basis are given by

$${}^{\epsilon}D^{\ell\,*}_{m0}(\phi,\theta,0) = \theta(m)\Big[D^{\ell\,*}_{m0}(\phi,\theta,0) - \epsilon(-)^m\, D^{\ell\,*}_{-m0}(\phi,\theta,0)\Big] \tag{A18}$$

It is seen that the modified D-functions are real if $\epsilon = -1$ and imaginary if $\epsilon = +1$:

$$\begin{aligned} {}^{(-)}D^{\ell\,*}_{m0}(\phi,\theta,0) &= 2\theta(m)d^{\ell}_{m0}(\theta)\cos m\phi \\ {}^{(+)}D^{\ell\,*}_{m0}(\phi,\theta,0) &= 2i\theta(m)d^{\ell}_{m0}(\theta)\sin m\phi \end{aligned} \tag{A19}$$

The overall amplitude in the reflectivity basis is now

$${}^{\epsilon}U_k(\Omega) = \sum_{\ell m} {}^{\epsilon}V_{\ell m k}\, {}^{\epsilon}A_{\ell m}(\Omega) \tag{A20}$$

where

$${}^{\epsilon}A_{\ell m}(\Omega) = \sqrt{\frac{2\ell+1}{4\pi}}\, {}^{\epsilon}D^{\ell\,*}_{m0}(\phi,\theta,0) \tag{A21}$$

and the resulting angular distribution is

$$I(\Omega) = \sum_{\epsilon k} |{}^{\epsilon}U_k(\Omega)|^2 \tag{A22}$$

It is seen that the sum involves four non-interfering terms for $\epsilon = \pm$ and $k = 1, 2$. The absence of the interfering terms of different reflectivities is a direct consequence of parity conservation in the production process. Following convention, we use the notation for partial amplitudes via

$$[\,\ell\,]_0 = {}^{(-)}V_{\ell 0}, \quad [\,\ell\,]_- = {}^{(-)}V_{\ell 1}, \quad [\,\ell\,]_+ = {}^{(+)}V_{\ell 1} \tag{A23}$$

where $[\,\ell\,]$ stands for the partial waves S, P, D, F and G for ℓ =0, 1, 2, 3 and 4.

Consider an example where the maximum ℓ is 2. One sees that there are in general twelve non-zero experimental moments:

$$\begin{aligned} &H(00),\ H(10),\ H(11),\ H(20),\ H(21),\ H(22) \\ &H(30),\ H(31),\ H(32),\ H(40),\ H(41),\ H(42) \end{aligned} \tag{A24}$$

while the partial waves $[\,\ell\,]$ are, for unnatural-parity exchange,

$$S_0, \quad P_0, \quad P_-, \quad D_0, \quad D_- \tag{A25}$$

and, for natural-parity exchange,

$$P_+, \quad D_+ \tag{A26}$$

One wave in each naturality can be set be real (S_0 and P_+, for example), so that there are again twelve real parameters (to be determined). It is helpful to write down the moments explicitly in terms of the partial waves:

$$
\begin{aligned}
H(00) &= S_0^2 + P_0^2 + P_-^2 + D_0^2 + D_-^2 + P_+^2 + D_+^2 \\
H(10) &= \frac{1}{\sqrt{3}} S_0 P_0 + \frac{2}{\sqrt{15}} P_0 D_0 + \frac{1}{\sqrt{5}} (P_- D_- + P_+ D_+) \\
H(11) &= \frac{1}{\sqrt{6}} S_0 P_- + \frac{1}{\sqrt{10}} P_0 D_- - \frac{1}{\sqrt{30}} P_- D_0 \\
H(20) &= \frac{1}{\sqrt{5}} S_0 D_0 + \frac{2}{5} P_0^2 - \frac{1}{5} (P_-^2 + P_+^2) + \frac{2}{7} D_0^2 + \frac{1}{7} (D_-^2 + D_+^2) \\
H(21) &= \frac{1}{\sqrt{10}} S_0 D_- + \frac{1}{5} \sqrt{\frac{3}{2}} P_0 P_- + \frac{1}{7\sqrt{2}} D_0 D_- \\
H(22) &= \frac{1}{5} \sqrt{\frac{3}{2}} (P_-^2 - P_+^2) + \frac{1}{7} \sqrt{\frac{3}{2}} (D_-^2 - D_+^2) \\
H(30) &= \frac{3}{7\sqrt{5}} (\sqrt{3} P_0 D_0 - P_- D_- - P_+ D_+) \\
H(31) &= \frac{1}{7} \sqrt{\frac{3}{5}} (2 P_0 D_- + \sqrt{3} P_- D_0) \\
H(32) &= \frac{1}{7} \sqrt{\frac{3}{2}} (P_- D_- - P_+ D_+) \\
H(40) &= \frac{2}{7} D_0^2 - \frac{4}{21} (D_-^2 + D_+^2) \\
H(41) &= \frac{1}{7} \sqrt{\frac{5}{3}} D_0 D_- \\
H(42) &= \frac{\sqrt{10}}{21} (D_-^2 - D_+^2)
\end{aligned}
\tag{A27}
$$

APPENDIX B: MAXIMUM-LIKELIHOOD ANALYSIS

This appendix is devoted to an exposition of the experimental moments, the acceptance moments and the acceptance-corrected (or 'true') moments and the relationships among them. Finally, the extended likelihood functions are given as functions of the 'true' and acceptance moments.

One may determine directly the experimental moments (unnormalized) as follows:

$$
H_x(LM) = \sum_i^n \; D^L_{M\,0}(\phi_i, \theta_i, 0) \tag{B1}
$$

where the sum is over a given number n of experimental data in a mass bin. But this is given by, from (A8),

$$H_x(LM) = \int \mathrm{d}\Omega\ \eta(\Omega)\, I(\Omega)\, D^L_{M0}(\phi,\theta,0) \tag{B2}$$

where $\eta(\Omega)$ represents the finite acceptance of the apparatus, and it includes software cuts, if any. From (A5), one finds that

$$H_x(LM) = \sum_{L'M'} H(L'M')\ \Psi_x(LM\, L'M') \tag{B3}$$

where

$$\Psi_x(LM\, L'M') = \left(\frac{2L'+1}{4\pi}\right) \int \mathrm{d}\Omega\ \eta(\Omega)\ D^L_{M0}(\phi,\theta,0)\, D^{L'\,*}_{M'0}(\phi,\theta,0) \tag{B4}$$

Note that the Ψ's have a simple normalization

$$\Psi_x(LM\, L'M') = \delta_{L\,L'}\delta_{M\,M'} \tag{B5}$$

in the limit $\eta(\Omega) = 1$. The integral (B4) can be calculated using a sample of 'accepted' MC events. Let N_x be the number of accepted MC events, out of a total of N raw MC events. Then, the integral is

$$\Psi_x(LM\, L'M') = \left(\frac{2L'+1}{4\pi}\right)\ \frac{1}{N}\sum_i^{N_x}\ D^L_{M0}(\phi_i,\theta_i,0)\, D^{L'\,*}_{M'0}(\phi_i,\theta_i,0) \tag{B6}$$

The equation (B3) shows that one can predict the experimentally measurable moments (B1), given a set $\{H\}$ and Ψ's; this provides one a means of assessing the goodness of fit by forming a χ^2 based on the set $\{H_x\}$.

There exists an alternative method of determining Ψ's. For the purpose, one expands the acceptance function $\eta(\Omega)$ in terms of the orthonormal D-functions, as follows:

$$\eta(\Omega) = \sum_{LM}(2L+1)\xi(LM)\, D^{L\,*}_{M0}(\phi,\theta,0) \tag{B7}$$

where $\xi(LM)$ is given by

$$\xi(LM) = \frac{1}{4\pi}\int \mathrm{d}\Omega\ \eta(\Omega)\, D^L_{M0}(\phi,\theta,0) \tag{B8}$$

The complex conjugate is, from the defining formula above,

$$\xi^*(LM) = (-)^M \, \xi(L \; -M) \tag{B9}$$

so that the acceptance function can be made explicitly real

$$\eta(\Omega) = \sum_{LM} (2L+1)\tau(M)\mathrm{Re}\left\{\xi(LM)\, D^{L\,*}_{M0}(\phi,\theta,0)\right\} \tag{B10}$$

where

$$\begin{aligned} \tau(M) &= 2, \quad M > 0, \\ &= 1, \quad M = 0, \\ &= 0, \quad M < 0 \end{aligned} \tag{B11}$$

One sees that $\tau(M)$ defined in equation (A13) $= 4\theta^2(M)$.

A set of $\xi(LM)$ specifies completely the acceptance in the problem. The normalization for the acceptance function has be chosen so that a perfect acceptance is given by $\eta(\Omega) = 1$ and $\xi(LM) = \delta_{L0}\delta_{M0}$. The $\xi(LM)$'s can be measured experimentally using the accepted MC events

$$\xi(LM) = \frac{1}{4\pi N} \sum_i^{N_x} \; D^L_{M0}(\phi_i, \theta_i, 0) \tag{B12}$$

Finally, substituting (B7) into (B4), one finds

$$\Psi_x(LM\,L'M') = \sum_{L''M''} (2L''+1)\xi^*(L''M'')(LM L''M''|L'M')(L0L''0|L'0) \tag{B13}$$

This formula shows an important aspect of the $\xi(LM)$ technique of representing acceptance. Although (B8) involves a sum in which L and M could be extended to infinity for an arbitrary acceptance, there is a cutoff if the set $\{H\}$ has maxima L_m and M_m [see (B3)]. The formula above demonstrates that $L'' \leq 2L_m$ and $|M''| \leq 2M_m$.

In a partial-wave analysis, it is usually best to take a set of the partial waves, $[\ell]_0$, $[\ell]_-$ and $[\ell]_+$, as unknown parameters to be determined in an extended maximum-likelihood fit. Since there is an absolute scale in an extended maximum-likelihood fit, one has then in possession the predicted numbers of events for all the partial waves, corrected for finite acceptance and angular distributions. The partial waves in turn give rise to a set of predicted moments $\{H\}$. But the $H(00)$ is not 1 but the total predicted number of events from the fit, i.e. one should be using the unnormalized moments. It is possible to choose H's as unknowns in the fit, but the two sets of H's should be the same ideally—this affords one an effective way of assessing self-consistency between the chosen moments and the partial waves.

For completeness, a short comment is given of the extended likelihood functions. The likelihood function for finding 'n' events of a given bin with a finite acceptance $\eta(\Omega)$ is defined as a product of the probabilities,

$$\mathcal{L} \propto \left[\frac{\bar{n}^n}{n!}\,\mathrm{e}^{-\bar{n}}\right] \prod_i^n \left[\frac{I(\Omega_i)}{\int I(\Omega)\,\eta(\Omega)\,d\Omega}\right] \tag{B14}$$

where the first bracket is the Poisson probability for 'n' events. This is the so-called extended likelihood function, in the sense that the Poisson distribution for 'n' itself is included in the likelihood function. Note that the expectation value $\bar{n}$ for n is given by

$$\bar{n} \propto \int I(\Omega)\,\eta(\Omega)\,d\Omega \tag{B15}$$

The likelihood function $\mathcal{L}$ can now be written, dropping the factors depending on n alone,

$$\mathcal{L} \propto \left[\prod_i^n I(\Omega_i)\right] \exp\left[-\int I(\Omega)\,\eta(\Omega)\,d\Omega\right]$$

The 'log' of the likelihood function now has the form,

$$\ln\mathcal{L} \propto \sum_i^n \ln I(\Omega_i) - \int \mathrm{d}\Omega\,\eta(\Omega)\,I(\Omega) \tag{B16}$$

which can be recast in terms of the $\xi(LM)$'s

$$\begin{aligned} \ln\mathcal{L} &\propto \sum_i^n \ln I(\Omega_i) - \sum_{LM} (2L+1)\,H(LM)\,\xi^*(LM) \\ &\propto \sum_i^n \ln I(\Omega_i) - \sum_{LM} (2L+1)\,\tau(M)\,H(LM)\,\mathrm{Re}\xi(LM) \end{aligned} \tag{B17}$$

$H(LM)$'s may be used directly as parameters in the fit or may be given as functions of the partial waves. It is interesting to note that the $\xi(LM)$'s for $L > L_m$ and $|M| > M_m$ are not needed in the likelihood fit. Note also that only the real part of the $\xi(LM)$'s are used in the fit.

It should be borne in mind that a set of the moments $\{H\}$ may not always be expressed in terms of the partial waves. This is clear if one examines the formulas (A27). Consider, for example, an angular distribution in which $H(10)$ is the only non-zero moment. But this moment is given by a set of interference terms involving even-odd partial waves. So at least one term cannot be zero—for example, the interference term involving S- and P-waves. But then neither $H(00)$ nor $H(20)$ can be zero, since both S- and P-waves are non-zero. One must conclude then that a χ^2 based on the set $\{H_x\}$ may not necessarily be zero identically.

APPENDIX C: RANK OF THE DENSITY MATRIX

An assumption needed for the partial-wave analysis is that the density matrix has rank 1, i.e. the spin amplitudes do not depend on the nucleon helicities. Our justification, so far, has been that the fitted partial waves are very reasonable, that these waves can be fitted with a very simple mass-dependent formula, that Pomeron-exchange amplitudes are in general independent of nucleon helicities, and so on....

The purpose of this Appendix is to point out that, under a simple model for mass dependence of the partial waves, it is possible to *prove* that the spin density matrix has rank 1. Suppose that one has found a satisfactory fit under a rank-1 assumption. One can then show that, even if the problem involves both spin-nonflip and spin-flip at the nucleon vertex—i.e. it appears to be a rank-2 problem—the spin density matrix in reality has a rank of 1. Although this note is based on the results of our $\eta\pi^-$ analysis, the derivation does not depend on the decay channels; the conclusions apply equally well to any decay channel, e.g. the 1^{-+} state at 1.6 GeV coupling to $\pi\rho$ (a further discussion on this point is given at the end of Section 3).

This note relies on some technicalities generally well known, and so they have been presented without attribution. The reader may wish to consult a number of preprints and/or papers, which deal with them in some detail [26,28,33–35].

1. Partial Waves Produced via Natural-parity Exchange

Consider our $\eta\pi^-$ system produced via natural-parity exchange. It consists of just two waves D_+ and P_+ in the $a_2(1320)$ region. Without loss of generality, the decay amplitudes [27] can be considered real, i.e.

$$\begin{aligned} A_{_D}(\Omega) &= \sqrt{\frac{5}{4\pi}}\,\sqrt{2}\,d^2_{10}(\theta)\,\sin\phi = -\sqrt{\frac{5}{4\pi}}\,\sqrt{3}\,\sin\theta\,\cos\theta\sin\phi \\ A_{_P}(\Omega) &= \sqrt{\frac{3}{4\pi}}\,\sqrt{2}\,d^1_{10}(\theta)\,\sin\phi = -\sqrt{\frac{3}{4\pi}}\,\sin\theta\,\sin\phi \end{aligned} \tag{C1}$$

Since one deals with the partial waves produced only by natural-parity exchange, one can drop the subscript '+' from the waves, and the angular distribution is simply given by

$$\begin{aligned} I(\Omega) &\propto |D\,A_{_D}(\Omega) + P\,A_{_P}(\Omega)|^2 \\ &\propto \left(\frac{3}{4\pi}\right)\left|\sqrt{5}D\cos\theta + P\right|^2\,\sin^2\theta\,\sin^2\phi \\ &\propto \left(\frac{3}{4\pi}\right)\left[\,5|D|^2\,\cos^2\theta + 2\sqrt{5}\,\Re\{D^*P\}\,\cos\theta + |P|^2\,\right]\,\sin^2\theta\,\sin^2\phi \end{aligned} \tag{C2}$$

The integration over the angles can be carried out easily, to obtain

$$\int I(\Omega)\,\mathrm{d}\Omega \propto |D|^2 + |P|^2 \tag{C3}$$

as expected.

The spin density matrix is given by

$$I(\Omega) \propto |D\,A_{_D}(\Omega) + P\,A_{_P}(\Omega)|^2 = \sum_{k,k'} \rho_{k,k'}\,A_k\,A^*_{k'} \tag{C4}$$

where $\{k,k'\} = \{1,2\}$ and '1' ('2') corresponds to D (P). From this definition, one sees that

$$\rho = \begin{pmatrix} |D|^2 & D\,P^* \\ D^*\,P & |P|^2 \end{pmatrix} \tag{C5}$$

One can work out the eigenvalues of this 2×2 matrix:

$$\lambda = \{|D|^2 + |P|^2,\ 0\} \tag{C6}$$

One of the two allowed eigenvalues is zero, i.e. the rank of this matrix is 1. This is the 'rank-1' assumption one makes to carry out the partial-wave analysis and is valid for a given mass bin.

Suppose now that the rank is 2, i.e.

$$I(\Omega) \propto |D_1\, A_{\scriptscriptstyle D}(\Omega) + P_1\, A_{\scriptscriptstyle P}(\Omega)|^2 + |D_2\, A_{\scriptscriptstyle D}(\Omega) + P_2\, A_{\scriptscriptstyle P}(\Omega)|^2 \tag{C7}$$

where subscripts 1 and 2 stand for spin-nonflip and spin-flip amplitudes at the nucleon vertex for reaction (1). Comparing (C2) and (C7), one finds immediately

$$\begin{aligned} |D|^2 &= |D_1|^2 + |D_2|^2 \\ |P|^2 &= |P_1|^2 + |P_2|^2 \\ \Re\{P^*\, D\} &= \Re\{P_1^*\, D_1\} + \Re\{P_2^*\, D_2\} \end{aligned} \tag{C8}$$

Let w be the effective mass of the $\eta\pi^-$ system. If the mass dependence is included explicitly in the formula, one should write, in the case of rank 1,

$$\frac{\mathrm{d}\sigma(w,\Omega)}{\mathrm{d}w\,\mathrm{d}\Omega} \propto |D(w)\, A_{\scriptscriptstyle D}(\Omega) + P(w)\, A_{\scriptscriptstyle P}(\Omega)|^2\ pq \tag{C9}$$

where p is the breakup momentum of the $\eta\pi^-$ system in the overall CM system and q is the breakup momentum of η in the $\eta\pi^-$ rest frame. Note that both p and q depend on w. Note also that the w dependence of the partial waves D and P are given in the formula. Obviously, a similar expression could be written down for the case of rank 2.

One is now ready to make the one crucial assumption for a mass-dependent analysis of the D and P waves: one assumes that two resonances—in D and P waves, respectively—are produced in *both* spin-nonflip and spin-flip amplitudes. One may then write, for the rank-1 case,

$$
\begin{aligned}
D(w) &= a\, e^{i\,\alpha}\, e^{i\,\delta_a}\, \sin\delta_a \\
P(w) &= b\, e^{i\,\delta_b}\, \sin\delta_b
\end{aligned}
\tag{C10}
$$

where a, b and the production phase α are all real and *independent* of the $\eta\pi^-$ mass. In addition, one can set $a \geq 0$ and $b \geq 0$ without loss of generality. δ_a and δ_b are the phase-shifts corresponding to the resonances and are highly mass dependent. In its generic form, the Breit-Wigner formula is given by the usual expression

$$
\cot\delta = \frac{w_0^2 - w^2}{w_0\,\Gamma_0}
\tag{C11}
$$

where w_0 and Γ_0 are the standard resonance parameters. In this note, the width is considered independent of w. Likewise, the barrier factor dependence for D and P is ignored.[k]

The formulas (C10) are generalized to the case of rank 2, as follows:

$$
\begin{aligned}
D_1(w) &= a_1\, e^{i\,\alpha_1}\, e^{i\,\delta_a}\, \sin\delta_a \\
P_1(w) &= b_1\, e^{i\,\delta_b}\, \sin\delta_b \\
D_2(w) &= a_2\, e^{i\,\alpha_2}\, e^{i\,\delta_a}\, \sin\delta_a \\
P_2(w) &= b_2\, e^{i\,\delta_b}\, \sin\delta_b
\end{aligned}
\tag{C12}
$$

Once again, a_i, b_i and α_i are real, $a_i \geq 0$ and $b_i \geq 0$, and *independent* of w. One finds, using (C8),

$$
\begin{aligned}
a^2 &= a_1^2 + a_2^2 \\
b^2 &= b_1^2 + b_2^2 \\
ab\cos(\alpha + \delta_a - \delta_b) &= a_1 b_1 \cos(\alpha_1 + \delta_a - \delta_b) + a_2 b_2 \cos(\alpha_2 + \delta_a - \delta_b)
\end{aligned}
\tag{C13}
$$

[k] Although simplified formulas are used in this note, the results given here do *not* change even when correct formulas are used. Note that, to go over to a correct formulation for each wave, one needs to substitute the absolute value of the Breit-Wigner formula as follows:

$$
\sin\delta(w) \to B(q)\left[\frac{\Gamma_0}{\Gamma(w)}\right]\sin\delta(w)
$$

where $B(q)$ is the barrier factor and $\Gamma(w)$ is the mass-dependent width. It should be noted that the correction factors are all real, by definition.

A plot of $\cos(\alpha + \delta_a - \delta_b)$ as a function of w is shown in Fig. 28 for three values of α, i.e. 0°, 45° and 90°. The resonance parameters for a and b of 1.0 and 0.151 are taken from the mass dependent fit of Section V B as are the resonant masses and widths.[l] The normalized absolute squares of the Breit-Wigner forms are given in Fig. 29, as is the 'normalized' interference term. The same quantities, as they appear in Ref. [1], are shown in Fig. 30. This figure shows how important the interference term is compared to the P-wave term. Note also how rapidly the interference term varies as a function of w in the $a_2(1320)$ region. This term, of course, is intimately related to the asymmetry in the Jackson angle and vanishes when integrated over the angle, i.e. it does not contribute to the mass spectrum [see (C2) and (C3)]. Fig. 31 shows the contour plot of the intensity distribution in w vs. $\cos\theta$; note the variation of the asymmetry as a function w.

For the last equation in (C13) to be true for any mass, the coefficient of $\cos(\delta_a - \delta_b)$ or $\sin(\delta_a - \delta_b)$ on the left-hand side must be equal to that on the right-hand side, so that

$$\begin{aligned} ab\cos\alpha &= a_1 b_1 \cos\alpha_1 + a_2 b_2 \cos\alpha_2 \\ ab\sin\alpha &= a_1 b_1 \sin\alpha_1 + a_2 b_2 \sin\alpha_2 \end{aligned} \tag{C14}$$

Taking the sum of the squares of the two formulas above and introducing the first two equations of (C13), one obtains:

$$\begin{aligned} &2a_1 b_1 a_2 b_2 \, \cos\alpha_1 \cos\alpha_2 + 2a_1 b_1 a_2 b_2 \, \sin\alpha_1 \sin\alpha_2 \\ =&a_1^2 b_2^2 + a_2^2 b_1^2 \\ =&a_1^2 b_2^2 \left(\cos^2\alpha_1 + \sin^2\alpha_1\right) + a_2^2 b_1^2 \left(\cos^2\alpha_2 + \sin^2\alpha_2\right) \end{aligned} \tag{C15}$$

which is recast into

$$0 = (a_1 b_2 \, \cos\alpha_1 - a_2 b_1 \, \cos\alpha_2)^2 + (a_1 b_2 \, \sin\alpha_1 - a_2 b_1 \, \sin\alpha_2)^2 \tag{C16}$$

It is clear that each term must be set to zero, so that

---

[l] The value of α as determined from this fit is 37.46°; for the purpose of illustration, one may consider $\alpha = 45°$ close enough.

$$
\begin{aligned}
\left(\frac{a_1}{b_1}\right)\cos\alpha_1 &= \left(\frac{a_2}{b_2}\right)\cos\alpha_2 \\
\left(\frac{a_1}{b_1}\right)\sin\alpha_1 &= \left(\frac{a_2}{b_2}\right)\sin\alpha_2
\end{aligned}
\tag{C17}
$$

Placing these back into (C14), one deduces that

$$
\begin{aligned}
\left(\frac{a}{b}\right)\cos\alpha = \left(\frac{a_1}{b_1}\right)\cos\alpha_1 &= \left(\frac{a_2}{b_2}\right)\cos\alpha_2 \\
\left(\frac{a}{b}\right)\sin\alpha = \left(\frac{a_1}{b_1}\right)\sin\alpha_1 &= \left(\frac{a_2}{b_2}\right)\sin\alpha_2
\end{aligned}
\tag{C18}
$$

One may take—alternately—the sum of the squares of the two formulas above, or a division of the second by the first, and obtain (remembering that the a's and b's are non-negative real quantites),

$$
\begin{aligned}
\frac{a}{b} = \frac{a_1}{b_1} &= \frac{a_2}{b_2} \\
\tan\alpha = \tan\alpha_1 &= \tan\alpha_2
\end{aligned}
\tag{C19}
$$

The last equation above demands that α_1 and α_2 are determined (up to $\pm\pi$), but they have to satisfy (C18). It is therefore clear that one must set $\alpha = \alpha_1 = \alpha_2$. Next, one introduces two new real variables $x \geq 0$ and $y \geq 0$, given by

$$
\begin{aligned}
x &= \frac{a_1}{a} = \frac{b_1}{b} \\
y &= \frac{a_2}{a} = \frac{b_2}{b}
\end{aligned}
\tag{C20}
$$

with the constraint $x^2 + y^2 = 1$.

Now one can prove that the case of rank 2 is reduced to that of rank 1. Indeed, one sees immediately that

$$
\begin{pmatrix} D_1 \\ P_1 \end{pmatrix} = x \begin{pmatrix} D \\ P \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} D_2 \\ P_2 \end{pmatrix} = y \begin{pmatrix} D \\ P \end{pmatrix}
\tag{C21}
$$

and (C7) becomes identical to (C2).

2. Discussion

It is shown in this Appendix that the problem of two resonances in D_+ and P_+ in the $\eta\pi^-$ system in (1) is—effectively—a rank-1 problem. For this to be true, the following conditions have to be met:

(a) There exist two distinct resonances with different masses and/or widths. Note that the crucial step, from (C13) to (C14), depends on that fact that $\delta_a - \delta_b$ is non-zero and is mass dependent.

(b) There exists a satisfactory rank-1 fit with two resonances in a given mass region, in which each amplitude for D_+ or P_+ has the following general form

$$\mathcal{M}_k(w,\Omega) = r_k\, \mathrm{e}^{i\,\alpha_k}\, \mathrm{e}^{i\,\delta_k(w)}\, f_k(w)\, A_k(\Omega) \tag{C22}$$

where $k = \{1,2\}$ and '1' ('2') corresponds to D_+ (P_+). $\delta_k(w)$ is the Breit-Wigner phase and highly mass dependent, while r_k and α_k are *mass independent* in the fit. Of course, one of the two α_k's can be set to zero without loss of generality, so that there are three independent parameters, e.g. r_1, r_2 and α_1 (these were denoted a, b and α, respectively, in the previous section). $f_k(w)$ contains the absolute value of the Breit-Wigner form, plus any other mass-dependent factors introduced in the model. $A_k(\Omega)$ carries the information about the rotational property of a partial wave k.

(c) The same two D_+ and P_+ resonances are produced in both spin-nonflip and spin-flip amplitudes, with the same general form as given above—but with arbitrary r_k's and α_k's for each spin-nonflip and spin-flip amplitude. It has been shown in this appendix that only one set of r_k's and α_k's, i.e. r_1, r_2 and α_1, is required for both spin-nonflip and spin-flip amplitudes. (This is indeed a remarkable result; the rank-2 problem entails a set of six parameters, but it has been shown that the set is reduced to that consisting of just three.) Therefore, the distribution function in both w and Ω is given by

$$\frac{\mathrm{d}\sigma(w,\Omega)}{\mathrm{d}w\,\mathrm{d}\Omega} \propto |\sum_k \mathcal{M}_k(w,\Omega)|^2\, pq \tag{C23}$$

independent of the nucleon helicities.

In another words, the spin density matrix has rank 1. The key ingredients for this result are that both spin-nonflip and spin-flip amplitudes harbor two resonances in D_+ and P_+ and that the production phase is mass-independent. It should be emphasized that the derivation given in this note does *not* depend on the existence of a good mass fit; it merely states that any fit with a mass-independent production phase is necessarily a rank-1 fit. Of course, the point is moot, if there exists no satisfactory fit in this model.

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FIGURES

FIG. 1. Experimental layout for E852.

FIG. 2. Average acceptance vs. effective mass.

FIG. 3. Acceptance vs. $cos\theta$ for different effective mass regions.

FIG. 4. Acceptance vs. ϕ for different effective mass regions.

FIG. 5. Acceptance vs. $|t'|$.

FIG. 6. Two-photon effective mass distribution.

FIG. 7. Missing-massed squared distribution.

FIG. 8. Missing-mass squared vs. two-photon mass.

FIG. 9. Effective-mass distribution of the background.

FIG. 10. Angular distribution of the background.

FIG. 11. $\eta\pi^-$ effective mass distribution.

FIG. 12. Distribution of $|t'| = |t - t_{\min}|$, where t is the four-momentum transfer.

FIG. 13. Distributions of the cosine of the decay angle in the GJ frame for various effective mass selections.

FIG. 14. Forward-backward asymmetry as a function of effective mass. The asymmetry $= (F - B)/(F + B)$ where F(B) is the number of events for which the η decays in the forward (backward) hemisphere in the GJ frame.

FIG. 15. Distributions of the Trieman-Yang angle ϕ in the GJ frame for various effective mass selections.

FIG. 16. Effective mass distributions for: a.) the $\pi^- p$; and the b.) ηp systems.

FIG. 17. Results of the partial wave amplitude analysis. Shown are a) the fitted intensity distributions for the D_+ and b) the P_+ partial waves, and c) their phase difference $\Delta\Phi$. The range of values for the eight ambiguous solutions is shown by the central bar and the extent of the maximum error is shown by the error bars. Also shown as curves in a), b), and c) are the results of the mass dependent analysis described in the text. The lines in d) correspond to (1) the fitted D_+ Breit-Wigner phase, (2) the fitted P_+ Breit-Wigner phase, (3) the fitted relative production phase ϕ, and (4) the overall phase difference $\Delta\Phi$.

FIG. 18. Results of the partial wave amplitude analysis. Shown are the fitted intensity distributions for the waves produced by unnatural-parity exchange.

FIG. 19. Forward-backward asymmetry as a function of effective mass. Shown are: the total asymmetry in the data (closed circles); the predicted asymmetry from the PWA fit (open squares); the prediction of the fit for that part of the asymmetry due to natural-parity exchange (filled squares); and the prediction of the fit for that part of the asymmetry due to the unnatural-parity exchange waves (open circles).

FIG. 20. Experimental moments (open circles) shown with the predicted moments (open triangles) from the amplitude analysis.

FIG. 21. Value of the log likelihood as a function of the number of P_+ events in the PWA fit for all 8 ambiguous solutions.

FIG. 22. Fitted P_+ intensity and $P_+ - D_+$ phase difference for the Monte Carlo sample generated with a pure D_+ sample of $a_2(1320)$ events.

FIG. 23. Comparison of the results of this amplitude analysis with the VES experiment.

FIG. 24. Comparison of the results of this amplitude analysis with those of the KEK experiment.

FIG. 25. The $\pi^+\pi^-\pi^0$ effective mass distribution for events with the topology of three forward charged tracks, one recoil charged track, and two photon clusters consistent with a π^0.

FIG. 26. The $\eta\pi^-$ effective mass distribution for the $\eta \to \pi^+\pi^-\pi^0$ event sample.

FIG. 27. Comparison of the results of the amplitude analysis for the $\eta \to \pi^+\pi^-\pi^0$ and the $\eta \to 2\gamma$ samples.

FIG. 28. $\cos(\alpha + \delta_a - \delta_b)$ as a function of w from 1.2 to 1.6 GeV for $\alpha = 0°(\diamond)$, $\alpha = 45°(+)$ and $\alpha = 90°(\boxdot)$.

FIG. 29. $\sin^2\delta_a$ $(\diamond)$, $\sin^2\delta_a$ $(+)$ and $\sin\delta_a \sin\delta_b \cos(\alpha + \delta_a - \delta_b)$ $(\boxdot)$ as a function of w from 1.2 to 1.6 GeV, using $\alpha = 45°$.

FIG. 30. $a^2 \sin^2\delta_a$ $(\diamond)$, $b^2 \sin^2\delta_a$ $(+)$ and $2\,a\,b \sin\delta_a \sin\delta_b \cos(\alpha + \delta_a - \delta_b)$ $(\boxdot)$ as a function of w from 1.2 to 1.6 GeV, where one has assumed that $a = 1.0$, $b = 0.20$ and $\alpha = 45°$.

FIG. 31. Angular distribution in $\cos\theta$ as a function of w from 1.2 to 1.6 GeV, where one has assumed that $a = 1.0$, $b = 0.151$ and $\alpha = 37.46°$.

TABLES

TABLE I. Comparison of the Results of E852 and the Crystal Barrel

	Mass MeV/c^2	Width MeV/c^2
E852	$1370 \pm 16\ ^{+50}_{-30}$	$385 \pm 40\ ^{+65}_{-105}$
Crystal Barrel	$1400 \pm 20 \pm 20$	$310 \pm 50\ ^{+50}_{-30}$